

Multidimensional Analytic Signals and the Bedrosian Identity*

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Abstract

The analytic signal method via the Hilbert transform is a key tool in signal analysis and processing, especially in the time-frequency analysis. Imaging and other applications to multidimensional signals call for extension of the method to higher dimensions. We justify the usage of partial Hilbert transforms to define multidimensional analytic signals from both engineering and mathematical perspectives. The important associated Bedrosian identity $T(fg) = fTg$ for partial Hilbert transforms T are then studied. Characterizations and several necessity theorems are established. We also make use of the identity to construct basis functions for the time-frequency analysis.

Keywords: multidimensional analytic signals, partial Hilbert transforms, the Bedrosian identity.

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1 Introduction

Signals are carrier of information. In many applications, feature information of a signal usually persists in the time-frequency domain [9]. The analytic signal method [10] is a classical way of defining without ambiguity the local amplitude and frequency of one-dimensional signals. Among others, it has been proven useful in meteorological and atmospheric applications, ocean engineering, structural science, and imaging processing, [13, 15, 16, 20, 21]. Especially, it motivates the widely used empirical mode decomposition and the Hilbert-Huang transform [7, 13, 14, 15].

The analytic signal method makes use of the Hilbert transform H defined for functions $f \in L^2(\mathbb{R})$ as

$$(Hf)(x) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{|y-x| \geq \varepsilon} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R}. \quad (1.1)$$

Let $\mathbb{H}^1(\mathbb{R})$ be the Banach space of all the functions $\phi \in L^1(\mathbb{R})$ such that $H\phi \in L^1(\mathbb{R})$ equipped with the norm

$$\|\phi\|_{\mathbb{H}^1(\mathbb{R})} := \|\phi\|_{L^1(\mathbb{R})} + \|H\phi\|_{L^1(\mathbb{R})}.$$

The Hilbert transform Hf of $f \in L^\infty(\mathbb{R})$ is defined as a BMO function that is in the dual of $\mathbb{H}^1(\mathbb{R})$ by

$$\int_{\mathbb{R}} (Hf)\phi dx = - \int_{\mathbb{R}} f H\phi dx, \quad \phi \in \mathbb{H}^1(\mathbb{R}).$$

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The *analytic signal* $\mathcal{A}f$ of a real $f \in L^2(\mathbb{R})$ is formed by adding to it its Hilbert transform as the imaginary part:

$$\mathcal{A}f := f + iHf. \quad (1.2)$$

With the amplitude-phase decomposition

$$(\mathcal{A}f)(t) = \rho(t)e^{i\theta(t)}, \quad t \in \mathbb{R},$$

the $\rho(t)$ and $\theta'(t)$ respectively are taken as the instantaneous amplitude and frequency of the signal f at time t .

The Hilbert transform in (1.2) has many favorable mathematical properties that account for the usefulness of the analytic signal method in the time-frequency analysis. Here we mention four of them. Let \mathbb{N} be the set of positive integers and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For a fixed $d \in \mathbb{N}$, we denote by $\|\cdot\|$ and (\cdot, \cdot) the standard Euclidean norm and inner product on \mathbb{R}^d . The *Schwartz class* $\mathcal{S}(\mathbb{R}^d)$ consists of infinitely differentiable functions φ on \mathbb{R}^d such that for all $p \in \mathbb{Z}_+$ and $q \in \mathbb{Z}_+^d$

$$\sup\{(1 + \|x\|^2)^p |f^{(q)}(x)| : x \in \mathbb{R}^d\} < +\infty.$$

The Fourier transform $\hat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is defined as

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^d} \varphi(x) e^{-i(x, \xi)} dx, \quad \xi \in \mathbb{R}^d,$$

which is again a function in $\mathcal{S}(\mathbb{R}^d)$. The Fourier transform can be extended to the space $\mathcal{S}'(\mathbb{R}^d)$ of temperate distributions on \mathbb{R}^d by a duality principle [11].

The Hilbert transform has an equivalent definition via the Fourier multiplier $-i \operatorname{sgn}$, where $\operatorname{sgn}(\xi)$ takes value $-1, 0, 1$ for $\xi < 0, \xi = 0$ and $\xi > 0$, respectively. Specifically, we have for all $f \in L^2(\mathbb{R})$

$$(\mathcal{H}f)^\wedge(\xi) = -i \operatorname{sgn}(\xi) f^\wedge(\xi), \quad \xi \in \mathbb{R}. \quad (1.3)$$

There are two important consequences of this. First of all, we have for all $f \in L^2(\mathbb{R})$ that

$$\operatorname{supp}(f + iHf) \subseteq \mathbb{R}_+ \quad (1.4)$$

where $\mathbb{R}_+ := [0, +\infty)$. In other words, the analytic signal suppresses all the negative frequency components of the original signal. This justifies the physical soundness of the analytic signal method. Secondly, the multiplier definition (1.3) makes it possible to develop fast algorithms making use of the lightning fast FFT to compute the analytic signal [18]. In addition to the above two properties, it was shown in [28, 29] that the Hilbert transform is the only continuous and homogeneous operator L such that

$$L(\cos(\omega_0 t + \phi_0)) = \sin(\omega_0 t + \phi_0), \quad \text{for all } \omega_0 > 0, \phi_0 \in \mathbb{R}.$$

This implies that the analytic signal method is the only one that satisfies some reasonable physical requirements [28, 29]. Finally, the Hilbert transform satisfies the Bedrosian theorem [1]: If $f, g \in L^2(\mathbb{R})$ satisfy either $\operatorname{supp} \hat{f} \subseteq \mathbb{R}_+, \operatorname{supp} \hat{g} \subseteq \mathbb{R}_+$ or $\operatorname{supp} \hat{f} \subseteq [-a, a], \operatorname{supp} \hat{g} \subseteq (-\infty, -a] \cup [a, \infty)$ for some positive number a , then there holds the *Bedrosian identity*

$$[H(fg)](x) = f(x)(Hg)(x), \quad x \in \mathbb{R}. \quad (1.5)$$

The Bedrosian identity simplifies the calculation of the Hilbert transform of a product of functions, helps understand the instantaneous amplitude and frequency of signals, and provides a method of constructing basic signals in the time-frequency analysis [1, 3, 4, 9, 19, 21, 23, 25, 36]. It has attracted

much interest from the mathematical community [6, 8, 22, 24, 30, 31, 33, 34, 35, 36]. Here, we mention an observation in [30]. It states that the Hilbert transform is essentially the only operator that satisfies the Bedrosian identity.

We conclude that the analytic signal (1.2) is justified by the above mathematical properties. This paper is motivated by the need [5, 12, 32] of defining multidimensional analytic signals for the time-frequency analysis of multidimensional signals. Naturally, we are inclined to define the analytic signal of $f \in L^2(\mathbb{R}^d)$ through a fixed operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ as $f + Tf$. Thus the definition reduces to the choice to the operator T . In references [5, 12], compositions of the partial Hilbert transforms are used. One of our purposes is to show that linear combinations of compositions of the partial Hilbert transforms are indeed the only choice from the viewpoint of similar mathematical properties introduced in the last paragraph. The justifications will be carried out in the next section. In Section 3, we investigate the multidimensional Bedrosian identity $T(fg) = fTg$ for $f, g \in L^2(\mathbb{R}^d)$, where T is a linear combination of compositions of the partial Hilbert transforms. In particular, a necessary and sufficient condition and the Bedrosian theorem will be established. The necessity of the Bedrosian theorem will be discussed as well. In Section 4, we construct basis multidimensional analytic signals by the results on the Bedrosian identity.

2 Multidimensional Analytic Signals

Set $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$ and $\mathbb{N}_n := \{1, 2, \dots, n\}$, $n \in \mathbb{N}$. The partial Hilbert transform H_j , $j \in \mathbb{N}_d$, is defined for $f \in L^2(\mathbb{R}^d)$ as

$$(H_j f)(x) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}^d} \frac{f(y)}{x_j - y_j} dy := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{|y_j - x_j| \geq \varepsilon} \frac{f(y)}{x_j - y_j} dy, \quad x \in \mathbb{R}^d.$$

It is known that H_j is a bounded linear operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, [26]. Moreover, there holds

$$(H_j f)^\wedge(\xi) = -i \operatorname{sgn}(\xi_j) f^\wedge(\xi), \quad \xi \in \mathbb{R}^d, \quad j \in \mathbb{N}_d. \quad (2.1)$$

The purpose of this section is to justify the using of linear combinations of compositions of the partial Hilbert transforms to define the multidimensional analytic signal. For simplicity, we denote by H_0 the identity operator on $L^2(\mathbb{R}^d)$. A linear combination T of compositions of the partial Hilbert transforms has the form

$$T = \sum_{\alpha \in \mathbb{Z}_{d+1}^d} c_\alpha \mathcal{H}_\alpha, \quad (2.2)$$

where $\mathcal{H}_\alpha := \prod_{j \in \mathbb{N}_d} H_{\alpha_j}$ and c_α are complex constants. We shall argue from four points of view that the multidimensional analytic signal $\mathcal{A}f$ of $f \in L^2(\mathbb{R}^d)$ should be defined as

$$\mathcal{A}f := f + Tf \quad (2.3)$$

via an operator T of the form (2.2).

2.1 Concentration of the Frequency

By (1.4), the negative Fourier frequency of a one-dimensional analytic signal is suppressed while the energy of the positive frequency is doubled. For some applications, it is desirable to restrict the frequency to a certain d -hyperoctant, especial the first one, [12]. Following the notations of [30], we let ν^k , $k \in \mathbb{N}_{2^d}$, be the extreme points of the cube $[-1, 1]^d$. Assume that ν^1 and ν^2 are the point each of

whose component is 1 and -1 , respectively. The Euclidean space \mathbb{R}^d is divided into 2^d d -hyperoctants as

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{N}_{2^d}} Q_k,$$

where $Q_k := \{\xi : \xi \in \mathbb{R}^d, \nu_j^k \xi_j \geq 0, j \in \mathbb{N}_d\}$. Suppose that we have chosen the k -th d -hyperoctant and desire that for each $f \in L^2(\mathbb{R}^d)$ its multidimensional analytic signal $\mathcal{A}f$ be such that

$$(\mathcal{A}f)^\wedge(\xi) = \begin{cases} 2^d \hat{f}(\xi), & \xi \in Q_k, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

The above equation has the following equivalent form

$$(\mathcal{A}f)^\wedge(\xi) = \hat{f}(\xi) \prod_{j \in \mathbb{N}_d} (1 + \nu_j^k \operatorname{sgn}(\xi_j)), \quad \xi \in \mathbb{R}^d.$$

By (2.1), the above equations holds true for all $f \in L^2(\mathbb{R}^d)$ if and only if \mathcal{A} is defined by (2.3) with T there being given as

$$T = \prod_{j \in \mathbb{N}_d} (H_0 + i\nu_j^k H_j) - H_0,$$

which satisfies (2.2).

2.2 Computational Advantages

As a consequence of (2.1), it can be seen that a bounded linear operator T has the form (2.2) if and only if there exists some function $m \in L^\infty(\mathbb{R}^d)$ that is constant on each d -hyperoctants such that

$$(Tf)^\wedge = m \hat{f}, \quad f \in L^2(\mathbb{R}^d). \quad (2.5)$$

In other words, operators (2.2) are given by a very simple Fourier multiplier. Therefore, efficient numerical algorithms based on fast multidimensional discrete Fourier transforms (see [2] and the references cited therein) can be developed for the computation of the multidimensional analytic signal (2.3).

2.3 Physical Requirements

Following [28, 29], we ask what continuous linear operators T preserve the class of complex sinusoids. To make this precise, we need to extend the definition of partial Hilbert transforms. For each $\alpha \in \mathbb{Z}_{d+1}^d$, we define $\mathcal{H}_\alpha : L^1(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ by setting for each $f \in L^1(\mathbb{R}^d)$ as

$$\langle \mathcal{H}_\alpha f, \varphi \rangle := \prod_{j \in \mathbb{N}_d} (-1)^{\min(1, \alpha_j)} \int_{\mathbb{R}^d} f \mathcal{H}_\alpha \varphi dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \quad (2.6)$$

where for ϕ in a locally convex space V and $u \in V^*$ $\langle u, \phi \rangle := u(\phi)$. Since \mathcal{H}_α is bounded from $\mathcal{S}(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, (2.6) is well-defined. Denote by $\mathbb{H}^1(\mathbb{R}^d)$ the Banach space of functions $f \in L^1(\mathbb{R}^d)$ such that $H_j f \in L^1(\mathbb{R}^d)$ endowed with the norm

$$\|f\|_{\mathbb{H}^1(\mathbb{R}^d)} := \sum_{j \in \mathbb{Z}_{d+1}} \|H_j f\|_{L^1(\mathbb{R}^d)}.$$

The dual of $\mathbb{H}^1(\mathbb{R}^d)$ is the space $\text{BMO}(\mathbb{R}^d)$ of locally integrable functions on \mathbb{R}^d with bounded mean oscillation [26], which embeds continuously into $\mathcal{S}'(\mathbb{R}^d)$. Note that $L^\infty(\mathbb{R}^d) \subseteq \text{BMO}(\mathbb{R}^d)$. It is clear that \mathcal{H}_α is bounded from $\mathbb{H}^1(\mathbb{R}^d)$ to $\mathbb{H}^1(\mathbb{R}^d)$ for each $\alpha \in \mathbb{Z}_{d+1}^d$. Thus, we define for $f \in \text{BMO}(\mathbb{R}^d)$ $\mathcal{H}_\alpha f$ again as a BMO function by

$$\langle \mathcal{H}_\alpha f, g \rangle := \prod_{j \in \mathbb{N}_d} (-1)^{\min(1, \alpha_j)} \langle f, \mathcal{H}_\alpha g \rangle, \quad g \in \mathbb{H}^1(\mathbb{R}^d). \quad (2.7)$$

For $\xi \in \mathbb{R}^d$, we set $\text{sgn}(\xi) := (\text{sgn}(\xi_j) : j \in \mathbb{N}_d)$ and shall characterize operators that preserve the class of complex sinusoids.

Proposition 2.1 *Suppose that T is a linear operator from $L^\infty(\mathbb{R}^d) \cup L^2(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ that is continuous in the topology of $\mathcal{S}'(\mathbb{R}^d)$ and satisfies for some function $\lambda : \mathbb{C}^d \rightarrow \mathbb{C}$ that*

$$Te^{i(\omega, \cdot)} = \lambda(\text{sgn}(\omega))e^{i(\omega, \cdot)}, \quad \text{for all } \omega \in \mathbb{R}^d. \quad (2.8)$$

Then T has the form (2.2).

Proof: Let $f \in L^2(\mathbb{R}^d)$. Since $\mathbb{S} := \text{span}\{e^{i(\omega, \cdot)} : \omega \in \mathbb{R}^d\}$ is dense in $\mathcal{S}'(\mathbb{R}^d)$, there exists a sequence $f_n \in \mathbb{S}$ that converges to f in $\mathcal{S}'(\mathbb{R}^d)$. By the continuity of T , Tf_n converges to Tf . Thus, $(Tf_n)^\wedge \rightarrow (Tf)^\wedge$. Denote for $\omega \in \mathbb{R}^d$ by δ_ω the delta distribution at ω . We observe that

$$(e^{i(\omega, \cdot)})^\wedge = (2\pi)^d \delta_\omega.$$

Consequently,

$$(Te^{i(\omega, \cdot)})^\wedge = (2\pi)^d \lambda(\text{sgn}(\omega)) \delta_\omega. \quad (2.9)$$

Let φ be an arbitrary function in $\mathcal{S}_k(\mathbb{R}^d) := \{\varphi \in \mathcal{S}(\mathbb{R}^d) : \text{supp } \varphi \subseteq Q_k\}$ for some $k \in \mathbb{N}_{2d}$. Then we see that $\lambda(\text{sgn}(\cdot))\varphi \in \mathcal{S}(\mathbb{R}^d)$. Assume that $f_n = \sum_{j \in \mathbb{N}_{k_n}} c_{nj} e^{i(\omega_{nj}, \cdot)}$. We get by (2.9) that

$$\langle (Tf_n)^\wedge, \varphi \rangle = \sum_{j \in \mathbb{N}_{k_n}} c_{nj} (2\pi)^d \lambda(\text{sgn}(\omega_{nj})) \varphi(\omega_{nj}) = \langle \hat{f}_n, \lambda(\text{sgn}(\cdot))\varphi \rangle.$$

Since

$$\lim_{n \rightarrow \infty} \langle (Tf_n)^\wedge, \varphi \rangle = \langle (Tf)^\wedge, \varphi \rangle \quad \text{and} \quad \lim_{n \rightarrow \infty} \langle \hat{f}_n, \lambda(\text{sgn}(\cdot))\varphi \rangle = \langle \hat{f}, \lambda(\text{sgn}(\cdot))\varphi \rangle,$$

we have for all $\varphi \in \mathcal{S}_k(\mathbb{R}^d)$, $k \in \mathbb{N}_{2d}$ that

$$\langle (Tf)^\wedge, \varphi \rangle = \langle \lambda(\text{sgn}(\cdot))\hat{f}, \varphi \rangle.$$

Since the linear span of $\cup_{k \in \mathbb{N}_{2d}} \mathcal{S}_k(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, the above equation remains true for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Therefore, $(Tf)^\wedge = \lambda(\text{sgn}(\cdot))\hat{f}$. We complete the proof by pointing out that a linear operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ has the form (2.2) if and only if it is defined by a Fourier multiplier that is constant on each Q_k , $k \in \mathbb{N}_{2d}$. \square

2.4 The Bedrosian Identity

Considering the importance of the Bedrosian identity in the time-frequency of one-dimensional signals, it is natural for us to expect a linear operator T in the definition of the multidimensional analytic signal (2.3) to satisfy a multidimensional Bedrosian identity. Assume that a bounded linear operator

$T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is engaged to define multidimensional analytic signal by (2.3). First of all, we shall require that T is translation invariant, namely, for all $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$

$$T(f(\cdot - x)) = (Tf)(\cdot - x).$$

The reason is that there is usually a delay in time of an input in signal processing. Thus we would like the processor T to have the property that a delay in the input will simply cause the same delay in the output.

Now we are seeking extensions of the Bedrosian theorem that a bounded linear translation invariant operator could possibly satisfy. We say that a bounded linear operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ satisfies the *type one Bedrosian theorem* if the Bedrosian identity

$$T(fg) = fTg \tag{2.10}$$

holds true for all $f, g \in \mathcal{S}(\mathbb{R}^d)$ that satisfy

$$\text{supp } \hat{f} \subseteq \prod_{j \in \mathbb{N}_d} [-a_j, a_j], \quad \text{supp } \hat{g} \subseteq \prod_{j \in \mathbb{N}_d} \mathbb{R} \setminus (-a_j, a_j), \quad \text{for some } a = (a_j \geq 0 : j \in \mathbb{N}_d). \tag{2.11}$$

The following results was proved in [30].

Lemma 2.2 *A bounded linear translation invariant operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ satisfies the type one Bedrosian theorem if and only if it is of the form (2.2).*

Thus, if we desire that the operator T satisfy the type one Bedrosian theorem then it must be a linear combination of the compositions of the partial Hilbert transforms. A more natural extension of the Bedrosian theorem is the *type two Bedrosian theorem*, that is, (2.10) holds true for all $f, g \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\text{supp } \hat{f} \subseteq B(0, R), \quad \text{supp } \hat{g} \subseteq \mathbb{R}^d \setminus B^o(0, R), \quad \text{for some } R \geq 0, \tag{2.12}$$

where $B(x_0, r) := \{x \in \mathbb{R}^d : \|x - x_0\| \leq r\}$ for $x_0 \in \mathbb{R}^d$ and $r > 0$, and $B^o(x_0, r)$ denotes its interior. When $d = 1$, the type one and type two Bedrosian theorems are identically the same. However, we shall prove that for $d \geq 2$ there exist no nontrivial operators that satisfy the type two Bedrosian theorem.

Theorem 2.3 *Let $d \geq 2$. Then there does not exist a nontrivial bounded linear translation invariant operator $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ that satisfies the type two Bedrosian theorem.*

Proof: Let $T : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be bounded linear and translation invariant. Assume that it satisfies the type two Bedrosian identity. Let f, g be arbitrary functions in $\mathcal{S}(\mathbb{R}^d)$ that satisfies (2.11). Let $R := \|a\|$. Then we observe that (2.12) holds true. By the assumption, the Bedrosian identity (2.10) holds. We conclude that (2.10) holds for all $f, g \in \mathcal{S}(\mathbb{R}^d)$ satisfying (2.11) for some $a := (a_j \geq 0 : j \in \mathbb{N}_d)$. By Lemma 2.2, T must be of the form (2.2). Thus, T is defined by (2.5) via a Fourier multiplier $m \in L^\infty(\mathbb{R}^d)$ that is constant in each d -hyperoctant. Let m_k be the constant value that m takes in Q_k , $k \in \mathbb{N}_{2^d}$.

Assume that T is neither the zero operator nor a multiple of the identity operator. In other words, $m_{k_1} \neq m_{k_2}$ for some $k_1, k_2 \in \mathbb{N}_{2^d}$. Since $d \geq 2$, there must also exist some $k_3 \in \mathbb{N}_{2^d}$ such that $m_{k_1} \neq m_{k_3}$ or $m_{k_2} \neq m_{k_3}$. There hence exists a pair $l_1, l_2 \in \mathbb{N}_{2^d}$ such that $m_{l_1} \neq m_{l_2}$ and $\nu_{j_0}^{l_1} \nu_{j_0}^{l_2} = 1$ for some $j_0 \in \mathbb{N}_d$. For notational simplicity, assume that $l_1 = 1$ and $j_0 = 1$. Since $l_2 \neq 1$, the set

$\{j \in \mathbb{N}_d : \nu_j^{l_2} = -1\}$ is nonempty. As a consequence, there exist positive constants ϵ, r_1, r_2 so that $\xi^1, \xi^2 \in \mathbb{R}^d$ defined by $\xi^1 := (1 - \frac{\epsilon}{2}, r_1, r_1, \dots, r_1)$ and

$$\xi_k^2 := \begin{cases} 1 - \epsilon, & k = 1, \\ \frac{r_1}{2}, & \nu_k^{l_2} = 1, \\ -r_2, & \nu_k^{l_2} = -1. \end{cases}$$

satisfy $\|\xi^1\| < 1$, $\|\xi^2\| > 1$ and $\|\xi^1 - \xi^2\| < 1$. Choose $r > 0$ so small that $B(\xi^1 - \xi^2, r) \subseteq Q_1 \cap B(0, 1)$ and $B(\xi^2, r) \subseteq Q_{l_2} \cap (\mathbb{R}^d \setminus B(0, 1))$. One can construct nonnegative $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that $\text{supp } \hat{f} \subseteq B(\xi^1 - \xi^2, r)$, $\text{supp } \hat{g} \subseteq B(\xi^2, r)$, $\hat{f}(\xi^1 - \xi^2) > 0$, and $\hat{g}(\xi^2) > 0$. Then we have that

$$\text{supp } \hat{f} \subseteq B(0, 1), \quad \text{supp } \hat{g} \subseteq \mathbb{R}^d \setminus B(0, 1).$$

Since T satisfies the type two Bedrosian theorem, (2.10) holds. Applying the Fourier transform to both sides of (2.10) yields that

$$\int_{\mathbb{R}^d} \hat{f}(\xi - \eta) \hat{g}(\eta) (m(\xi) - m(\eta)) d\eta = 0, \quad \text{for all } \xi \in \mathbb{R}^d,$$

which has the following form at $\xi = \xi^1$,

$$(m_1 - m_{l_2}) \int_{B(\xi^2, r)} \hat{f}(\xi^1 - \eta) \hat{g}(\eta) d\eta = 0.$$

However, since $m_1 \neq m_{l_2}$, $\hat{f}(\xi^1 - \xi^2) \hat{g}(\xi^2) > 0$, and $\hat{f}(\xi^1 - \cdot) \hat{g}$ is nonnegative,

$$(m_1 - m_{l_2}) \int_{B(\xi^2, r)} \hat{f}(\xi^1 - \eta) \hat{g}(\eta) d\eta \neq 0.$$

We hence come to a contradiction and complete the proof. \square

Based on the above mathematical considerations, we conclude that the only operators that should be used in (2.3) to define the multidimensional analytic signal should be linear combinations of the compositions of the partial Hilbert transforms.

3 Multidimensional Bedrosian Identities

The purpose of this section is to study the Bedrosian identity (2.10) for the time-frequency analysis of multidimensional analytic signals. By the discussion in the last section, we shall assume from now on that T is of the form (2.2). In other words, there exist constants $m_k \in \mathbb{C}$, $k \in \mathbb{N}_{2d}$ such that T is defined by (2.5) through the Fourier multiplier $m \in L^\infty(\mathbb{R}^d)$ given as

$$m(\xi) := m_k, \quad \xi \in Q_k, \quad k \in \mathbb{N}_{2d}. \quad (3.1)$$

Denote by T^* the adjoint operator of T on $L^2(\mathbb{R}^d)$, that is,

$$(Tf, g)_{L^2(\mathbb{R}^d)} = (f, T^*g)_{L^2(\mathbb{R}^d)}, \quad f, g \in L^2(\mathbb{R}^d),$$

where $(\cdot, \cdot)_{L^2(\mathbb{R}^d)}$ is the inner product on $L^2(\mathbb{R}^d)$. It is clear that T^* is defined by the Fourier multiplier \overline{m} . By definition (2.6), as a linear combination of compositions of partial Hilbert transform, T is well-defined on $L^1(\mathbb{R}^d)$ with range in $\mathcal{S}'(\mathbb{R}^d)$ by

$$\langle Tf, \varphi \rangle = \langle f, \overline{T^* \varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \quad (3.2)$$

We shall start the characterization of the Bedrosian identity with a lemma on a property of the operator T . To this end, we recall that the Fourier transform \hat{S} of a temperate distribution $S \in \mathcal{S}'(\mathbb{R}^d)$ is again a temperate distribution satisfying

$$\langle \hat{S}, \varphi \rangle = \langle S, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \quad (3.3)$$

Lemma 3.1 *For all $f \in L^1(\mathbb{R}^d)$, $(Tf)^\wedge = m\hat{f}$.*

Proof: Let $f \in L^1(\mathbb{R}^d)$. We proceed by equation (3.3) and (3.2) that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\langle \hat{T}f, \varphi \rangle = \langle Tf, \hat{\varphi} \rangle = \langle f, \overline{T^*\hat{\varphi}} \rangle. \quad (3.4)$$

Set $h := (\overline{T^*\hat{\varphi}})^\vee$, the inverse Fourier transform of $\overline{T^*\hat{\varphi}}$. Note that $h \in L^1(\mathbb{R}^d)$. Thus, by (3.4)

$$\langle (Tf)^\wedge, \varphi \rangle = \int_{\mathbb{R}^d} f\hat{h}dx = \int_{\mathbb{R}^d} \hat{f}hdx..$$

We compute that

$$h = (\overline{T^*\hat{\varphi}})^\vee = \frac{1}{(2\pi)^d} \overline{(T^*\hat{\varphi})^\wedge} = \overline{(T^*(\hat{\varphi}))^\wedge} = \overline{m\hat{\varphi}} = m\varphi.$$

Combining the above two equations proves the lemma. \square

Theorem 3.2 *Let T be defined by the Fourier multiplier (3.1). Then $f, g \in L^2(\mathbb{R}^d)$ satisfies the Bedrosian identity (2.10) if and only if*

$$\sum_{j \in \mathbb{N}_{2^d}, j \neq k} (m_k - m_j) \int_{Q_j} \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta = 0, \quad \xi \in Q_k, \quad k \in \mathbb{N}_{2^d}. \quad (3.5)$$

Proof: Since the Fourier transform is injective from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, (2.10) holds if and only if

$$(T(fg))^\wedge = (fTg)^\wedge.$$

By Lemma 3.1 and writing the Fourier transform of a product of two functions as their convolution, the above equations is equivalent to

$$m(\xi) \int_{\mathbb{R}^d} \hat{f}(\xi - \eta) g(\eta) d\eta = \int_{\mathbb{R}^d} \hat{f}(\xi - \eta) m(\eta) \hat{g}(\eta) d\eta, \quad \text{a.e. } \xi \in \mathbb{R}^d. \quad (3.6)$$

Since the left hand side above is continuous about $\xi \in \mathbb{R}^d$, (3.6) holds if and only if

$$m(\xi) \int_{\mathbb{R}^d} \hat{f}(\xi - \eta) g(\eta) d\eta = \int_{\mathbb{R}^d} \hat{f}(\xi - \eta) m(\eta) \hat{g}(\eta) d\eta, \quad \text{for all } \xi \in \mathbb{R}^d. \quad (3.7)$$

By (3.1), (3.7) has the form (3.5). The proof is complete. \square

In the one-dimensional case, various characterizations of the Bedrosian identity have been proposed in the literature. For instance, the above theorem for $d = 1$ was proved in [36]. A similar result was presented in [3] under the assumptions that $f, g \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Other characterizations can be found in [33, 35].

We now turn to sufficient conditions for the Bedrosian identity. We first prove the following generalization of the Bedrosian theorem, which was first proved in [25] for the case when T is the *total Hilbert transform* $\prod_{j \in \mathbb{N}_d} H_j$.

Proposition 3.3 *If $f, g \in L^2(\mathbb{R}^d)$ satisfy for some $a := (a_j \geq 0 : j \in \mathbb{N}_d)$ and $b := (b_j \geq 0 : j \in \mathbb{N}_d)$ that*

$$\text{supp } \hat{f} \subseteq \prod_{j \in \mathbb{N}_d} [-a_j, b_j], \quad \text{supp } \hat{g} \subseteq \prod_{j \in \mathbb{N}_d} \mathbb{R} \setminus (-b_j, a_j) \quad (3.8)$$

then the Bedrosian identity (2.10) holds true.

Proof: It suffices to prove that the integrand in each integral of (3.5) vanishes almost everywhere. Let $\xi \in Q_k$ and $j \neq k \in \mathbb{N}_{2d}$. We need to show that $\hat{f}(\xi - \eta)\hat{g}(\eta)$ is zero for almost every $\eta \in Q_j$. Assume that there exists $\eta \in Q_j$ that is in the support of $\hat{f}(\xi - \cdot)\hat{g}$. Then $\eta \in \text{supp } \hat{g} \cap Q_j$ and $\xi - \eta \in \text{supp } \hat{f}$. Since $j \neq k$, there exists $l \in \mathbb{N}_d$ such that $\xi_l \eta_l \leq 0$. We may assume that $\xi_l \geq 0$ and $\eta_l \leq 0$. By (3.8), $\eta_l < -b_l$ while $\xi_l - \eta_l \leq b_l$, which is impossible. The contradiction completes the proof. \square

To present the next sufficient condition, we set for $j \in \mathbb{N}_{2d}$, $I_j := \{k \in \mathbb{N}_{2d} : m_k = m_j\}$ and $Q_j := \cup_{k \in I_j} Q_k$. We also call a subset $A \subseteq \mathbb{R}^d$ closed under addition if for all $x, y \in A$, $x + y \in A$.

Proposition 3.4 *Let $j \in \mathbb{N}_{2d}$. If $\text{supp } \hat{f} \cup \text{supp } \hat{g} \subseteq Q_j$ and Q_j is closed under addition then $f, g \in L^2(\mathbb{R}^d)$ satisfy the Bedrosian identity (2.10).*

Proof: By Theorem 3.8, it suffices to show that for all $\xi \in \mathbb{R}^d$

$$\int_{Q_j} \hat{f}(\xi - \eta)\hat{g}(\eta)(m(\xi) - m_j)d\eta = 0. \quad (3.9)$$

Clearly, the above equation holds true for $\xi \in Q_j$. Assume that there exists $\xi \in \mathbb{R}^d \setminus Q_j$ that does not satisfy (3.9). Thus, there must exist $\eta \in Q_j$ such that $\xi - \eta \in \text{supp } \hat{f} \subseteq Q_j$. Since Q_j is closed under addition, $\xi - \eta \in Q_j$ and $\eta \in Q_j$ imply that $\xi \in Q_j$, a contradiction. \square

In the one-dimensional case, the Bedrosian theorem has an appealing physical interpretation. Namely, it states that if $f \in L^2(\mathbb{R})$ has low Fourier frequency and $g \in L^2(\mathbb{R})$ has high Fourier frequency then they satisfy the Bedrosian identity (1.5). Similarly, condition (3.8) can be interpreted as that f has low Fourier frequency in each coordinate while g is of high Fourier frequency in each coordinate. It had been conjectured that the condition in the Bedrosian theorem was necessary for the one-dimensional Bedrosian identity (1.5) until an explicit contradicting example was constructed in [36]. We recall that

$$f(t) := \frac{1}{1+t^2}, \quad g(t) := \frac{1-2t^2}{4+5t^2+t^4}, \quad t \in \mathbb{R}$$

satisfy $H(fg) = fHg$ while $\text{supp } \hat{f} = \text{supp } \hat{g} = \mathbb{R}$. Using this example, we define

$$F(x) := \prod_{j \in \mathbb{N}_d} f(x_j), \quad G(x) := \prod_{j \in \mathbb{N}_d} g(x_j), \quad x \in \mathbb{R}^d.$$

Since T is of the form (2.2), $T(FG) = F(TG)$. This together with $\text{supp } \hat{F} = \text{supp } \hat{G} = \mathbb{R}^d$ implies that the condition (3.8) is also unnecessary for the multidimensional Bedrosian identity (2.10).

Surprisingly, a necessity about the one-dimensional Bedrosian theorem was obtained in [35]. It asserts that if $f, g \in L^2(\mathbb{R})$ satisfy the Bedrosian identity (1.5), $\text{supp } \hat{f} \subseteq [a, b]$ for some $a, b \geq 0$, and endpoints a, b are indeed contained in $\text{supp } \hat{f}$ then $\text{supp } \hat{g} \subseteq \mathbb{R} \setminus (-b, a)$. We shall present an extension of this result for the multidimensional identity (2.10). To this end, we recall that the convolution $\varphi * \psi$ of $\varphi, \psi \in L^2(\mathbb{R}^d)$ is defined by

$$(\varphi * \psi)(x) := \int_{\mathbb{R}^d} \varphi(x-t)\psi(t)dt, \quad x \in \mathbb{R}^d.$$

It is well-known that $\text{supp } \varphi * \psi \subseteq \text{supp } \varphi + \text{supp } \psi$. In the case that both φ, ψ are compactly supported, we have the celebrated Titchmarsh convolution theorem [17, 27].

For each subset $A \subseteq \mathbb{R}^d$, we denote by $\text{conv } A$ the convex hull of A in \mathbb{R}^d .

Lemma 3.5 *If both $\varphi, \psi \in L^2(\mathbb{R}^d)$ are compactly supported then*

$$\text{conv supp } (\varphi * \psi) = \text{conv supp } \varphi + \text{conv supp } \psi.$$

Theorem 3.6 *Let ν be an extreme point of $[-1, 1]^d$, $Q := \{\xi \in \mathbb{R}^d : \xi_j \nu_j \geq 0, j \in \mathbb{N}_d\}$, $Q' := -Q$, $m|_Q \neq m|_{Q'}$, and $f, g \in L^2(\mathbb{R}^d)$ satisfy $\text{supp } \hat{f}, \text{supp } \hat{g} \subseteq Q \cup Q'$. If $a := (a_j > 0 : j \in \mathbb{N}_d)$ and $b := (b_j > 0 : j \in \mathbb{N}_d)$ that $(a_j \nu_j : j \in \mathbb{N}_d), (-b_j \nu_j : j \in \mathbb{N}_d) \in \text{supp } \hat{f}$ and*

$$\text{supp } \hat{f} \subseteq \{\xi \in Q : \xi_j \nu_j \leq a_j, j \in \mathbb{N}_d\} \cup \{\xi \in -Q : -\xi_j \nu_j \leq b_j, j \in \mathbb{N}_d\},$$

then $g \in L^2(\mathbb{R}^d)$ satisfies the Bedrosian identity (2.10) if and only if

$$\text{supp } \hat{g} \subseteq \left\{ \xi \in Q : \sum_{j \in \mathbb{N}_d} \frac{\nu_j \xi_j}{b_j} \geq 1 \right\} \cup \left\{ \xi \in -Q : \sum_{j \in \mathbb{N}_d} \frac{-\nu_j \xi_j}{a_j} \geq 1 \right\}.$$

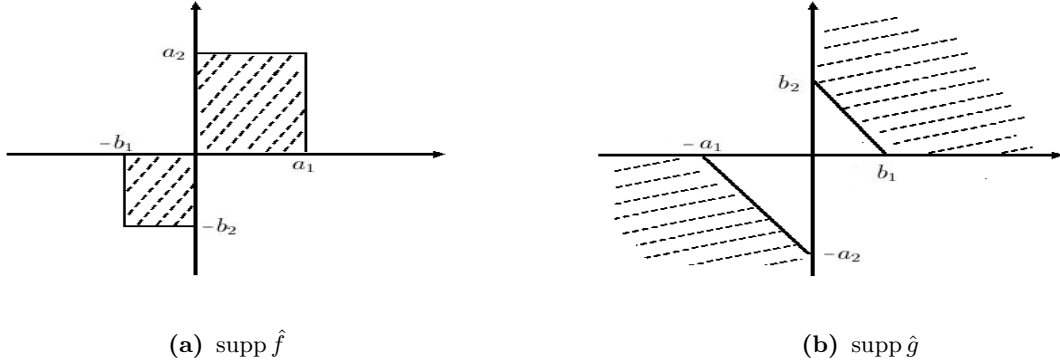
Before moving on to the proof, let us understand the conditions in Theorem 3.6. Suppose that $\nu = \nu^1$. Consequently, $Q = Q_1$ and $Q' = Q_2$. Theorem 3.6 states that given $f, g \in L^2(\mathbb{R}^d)$ with $\text{supp } \hat{g} \subseteq Q_1 \cup Q_2$, if

$$\text{supp } \hat{f} \subseteq \prod_{j \in \mathbb{N}_d} [0, a_j] \cup \prod_{j \in \mathbb{N}_d} [-b_j, 0], \text{ for some } a = (a_j > 0 : j \in \mathbb{N}_d), b = (b_j > 0 : j \in \mathbb{N}_d)$$

and $a, -b$ are actually contained in $\text{supp } \hat{f}$ then g satisfies the Bedrosian identity if and only if any $\xi \in \text{supp } \hat{g}$ lies in

$$\left\{ \xi : \xi_j \geq 0, \sum_{j \in \mathbb{N}_d} \frac{\xi_j}{b_j} \geq 1 \right\} \text{ or } \left\{ \xi : \xi_j \leq 0, \sum_{j \in \mathbb{N}_d} \frac{\xi_j}{a_j} \leq -1 \right\}.$$

We illustrate the supports of \hat{f} and \hat{g} for the two-dimensional case in the following graph.



We next present the proof of Theorem 3.6.

Proof: Without loss of generality, assume that $Q = Q_1$ and $Q' = Q_2$. By the assumptions that $\text{supp } \hat{f}, \text{supp } \hat{g} \subseteq Q_1 \cup Q_2$ and $m_1 \neq m_2$, we obtain from Theorem 3.2 that g satisfies the Bedrosian identity (2.10) if and only if

$$\int_{Q_2} \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta = 0, \quad \xi \in Q_1 \quad (3.10)$$

and

$$\int_{Q_1} \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta = 0, \quad \xi \in Q_2.$$

We shall only prove that $\text{supp}(\hat{g}\chi_{Q_2}) \subseteq \{\xi \in Q_2 : \sum_{j \in \mathbb{N}_d} \frac{\xi_j}{a_j} \leq -1\}$ since the other inclusion that $\text{supp}(\hat{g}\chi_{Q_1}) \subseteq \{\xi \in Q_1 : \sum_{j \in \mathbb{N}_d} \frac{\xi_j}{b_j} \geq 1\}$ can be handled in a similar way. Set $\varphi := \hat{f}\chi_{Q_1}$ and $\psi := \hat{g}\chi_{Q_2}$. We decompose ψ uniquely into

$$\psi = \psi_1 + \psi_2$$

where $\text{supp} \psi_1 \subseteq \prod_{j \in \mathbb{N}_d} [-a_j, 0]$ and $\text{supp} \psi_2 \subseteq Q_2 \setminus \prod_{j \in \mathbb{N}_d} (-a_j, 0)$. By equation (3.10),

$$(\varphi * \psi_1)(\xi) = -(\varphi * \psi_2)(\xi), \quad \xi \in Q_1.$$

Since

$$\text{supp}(\varphi * \psi_2) \subseteq \text{supp} \varphi + \text{supp} \psi_2 \subseteq Q_2 \setminus \prod_{j \in \mathbb{N}_d} [-a_j, 0] + \prod_{j \in \mathbb{N}_d} [0, a_j] \subseteq \mathbb{R}^d \setminus Q_1^o,$$

where Q_1^o is the interior of Q_1 , we have that

$$\text{supp}(\varphi * \psi_1) \subseteq \mathbb{R}^d \setminus Q_1^o.$$

This together with

$$\text{supp}(\varphi * \psi_1) \subseteq \text{supp} \varphi + \text{supp} \psi \subseteq \prod_{j \in \mathbb{N}_d} [-a_j, a_j]$$

implies that

$$\text{conv}(\varphi * \psi_1) \subseteq \{\xi : -a_j \leq \xi_j \leq a_j, \quad j \in \mathbb{N}_d, \quad \sum_{j \in \mathbb{N}_d} \frac{\xi_j}{a_j} \leq d-1\}. \quad (3.11)$$

We claim that for any $\xi \in \text{supp} \psi_1$, $\sum_{j \in \mathbb{N}_d} \frac{\xi_j}{a_j} \leq -1$. Assume to the contrary that this is invalid. Then there exists $\xi^0 \in \text{conv} \text{supp} \psi_1$ such that $\sum_{j \in \mathbb{N}_d} \frac{\xi_j^0}{a_j} > -1$. Note that $\xi^1 := a \in \text{supp} \varphi$, which satisfies $\sum_{j \in \mathbb{N}_d} \frac{\xi_j^1}{a_j} = d$. By Lemma 3.5,

$$\text{conv} \text{supp}(\varphi * \psi_1) = \text{conv} \text{supp} \psi_1 + \text{conv} \text{supp} \varphi.$$

It follows that there exists some point $\xi \in \text{conv} \text{supp}(\varphi * \psi_1)$ such that

$$\sum_{j \in \mathbb{N}_d} \frac{\xi_j}{a_j} > d-1,$$

contradicting (3.11). We conclude that

$$\text{supp} \psi_1 = \text{supp}(\hat{g}\chi_{Q_2}) \subseteq \{\xi \in Q_2 : \sum_{j \in \mathbb{N}_d} \frac{\xi_j}{a_j} \leq -1\},$$

which completes the proof. \square

By a direct application of the Titchmarsh convolution theorem, one obtains another necessity theorem analog to the one-dimensional one proved in [35].

Theorem 3.7 *Let $f \in L^2(\mathbb{R}^d)$ satisfy for some $a := (a_j \geq 0 : j \in \mathbb{N}_d)$ and $b := (b_j \geq 0 : j \in \mathbb{N}_d)$ that*

$$\text{supp } \hat{f} \subseteq \prod_{j \in \mathbb{N}_d} [-a_j, b_j]$$

and all the extreme points of $\prod_{j \in \mathbb{N}_d} [-a_j, b_j]$ are contained in $\text{supp } \hat{f}$. Then $g \in L^2(\mathbb{R}^d)$ with \hat{g} being compactly supported satisfies the Bedrosian identity $T(fg) = fTg$ for all operators T of the form (2.2) if and only if

$$\text{supp } \hat{g} \subseteq \prod_{j \in \mathbb{N}_d} \mathbb{R} \setminus (-b_j, a_j). \quad (3.12)$$

Proof: The sufficiency has been proved in Proposition 3.3. One sees that g satisfies $T(fg) = fTg$ for all operators T given by (2.2) if and only if for all $j \in \mathbb{N}_d$

$$H_j(fg) = fH_j(g). \quad (3.13)$$

Set $\mathbb{R}_{j+}^d := \{\xi \in \mathbb{R}^d : \xi_j \geq 0\}$ and $\mathbb{R}_{j-}^d := \{\xi \in \mathbb{R}^d : \xi_j \leq 0\}$. Introduce for each $\varphi \in L^2(\mathbb{R}^d)$ the associated pair of functions defined by

$$(\varphi_{j+})^\wedge = \begin{cases} \hat{\varphi}(\xi), & \xi \in \mathbb{R}_{j+}^d, \\ 0, & \text{elsewhere,} \end{cases} \quad (\varphi_{j-})^\wedge = \begin{cases} \hat{\varphi}(\xi), & \xi \in \mathbb{R}_{j-}^d, \\ 0, & \text{elsewhere,} \end{cases}$$

By Theorem 3.2, identity (3.13) implies

$$\text{supp } (\hat{f}_{j+} * \hat{g}_{j-}) \subseteq R_{j-}^d, \quad \text{supp } (\hat{f}_{j-} * \hat{g}_{j+}) \subseteq R_{j+}^d.$$

By Lemma 3.5, we get

$$\text{supp } (\hat{f}_{j+}) + \text{supp } (\hat{g}_{j-}) \subseteq R_{j-}^d, \quad \text{supp } (\hat{f}_{j-}) + \text{supp } (\hat{g}_{j+}) \subseteq R_{j+}^d.$$

Since the extreme points of $\prod_{j \in \mathbb{N}_d} [-a_j, b_j]$ lie in $\text{supp } \hat{f}$, there exist $\xi \in \text{supp } \hat{f}_{j+}$ and $\eta \in \hat{f}_{j-}$ with $\xi_j = b_j$ and $\eta_j = -a_j$. This together with the above equation imply

$$\text{supp } \hat{g} \subseteq \{\xi \in \mathbb{R}^d : \xi_j \in \mathbb{R} \setminus (-b_j, a_j)\}.$$

Since this is true for each $j \in \mathbb{N}_d$, we get (3.12). □

4 Basic Multidimensional Analytic Signals

Assume that an operator T of the form (2.2) has been chosen to define multidimensional analytic signals. Applying T directly to a given signal would generally not yield physically meaningful results for the reason that the signal may contain multiple components. It is desirable to first decompose the signal into a sum of basic signals that behave well under T , and then apply the operator T to each summand. Note that T given by (2.2) is a linear combination of compositions of partial Hilbert transforms. Thus, the purpose of this section is to construct basic signals that will behave well under each partial Hilbert transforms. More precisely, we shall first construct one-dimensional signals in

$$\mathcal{M} := \{\rho \cos \theta : \rho \in L^2(\mathbb{R}), \theta \in C^1(\mathbb{R}), \rho \geq 0, \theta' \geq 0, H(\rho \cos \theta) = \rho \sin \theta\}.$$

Multidimensional basic signals can then be formed by tensor products of functions in \mathcal{M} .

Constructions of functions $\rho \cos \theta$ in \mathcal{M} has recently been considered in [23], where the phase function θ was selected to be a strictly increasing function on \mathbb{R} . Consequently, the constructed signal $\rho \cos \theta$ does not possess much fluctuation, which, on the other hand, is usually required in the time-frequency analysis. Thus, we shall adopt a different approach by choosing periodic phase functions that still enjoy nonnegative derivative. Specifically, we shall use phase functions determined by a finite Blaschke product.

Set $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathbf{a} := (a_j : j \in \mathbb{N}_n) \in \mathbb{U}^n$. The finite Blaschke product $\mathcal{B}_{\mathbf{a}}$ associated with \mathbf{a} is a holomorphic function on \mathbb{U} defined as

$$\mathcal{B}_{\mathbf{a}}(z) := \prod_{j \in \mathbb{N}_n} \frac{z - a_j}{1 - \bar{a}_j z}, \quad z \in \mathbb{U}.$$

We let $\theta_{\mathbf{a}}$ be the phase function determined by $\mathcal{B}_{\mathbf{a}}$ as

$$e^{i\theta_{\mathbf{a}}(t)} = \mathcal{B}_{\mathbf{a}}(e^{it}), \quad t \in \mathbb{R},$$

and aim at characterizing all real functions $f \in L^2(\mathbb{R})$ such that

$$H(f \cos \theta_{\mathbf{a}}) = f \sin \theta_{\mathbf{a}}. \quad (4.1)$$

We start with a straightforward observation.

Lemma 4.1 *A real function $f \in L^2(\mathbb{R})$ satisfies (4.1) if and only if*

$$H(fe^{i\theta_{\mathbf{a}}}) = -ife^{i\theta_{\mathbf{a}}}. \quad (4.2)$$

Proof: Let $f \in L^2(\mathbb{R})$ be real. Assume that (4.1) holds true then we apply H to both sides of (4.1) to get that

$$-f \cos \theta_{\mathbf{a}} = H(f \sin \theta_{\mathbf{a}}).$$

Combining the above equation with (4.1) yields that

$$H(fe^{i\theta_{\mathbf{a}}}) = H(f \cos \theta_{\mathbf{a}}) + iH(f \sin \theta_{\mathbf{a}}) = f \sin \theta_{\mathbf{a}} - if \cos \theta_{\mathbf{a}} = -ife^{i\theta_{\mathbf{a}}},$$

which is (4.2). Conversely, if (4.2) holds true then by comparing the real part of its both sides, we obtain (4.1). Thus, it suffices to show that (4.2) is equivalent to (4.3). \square

We first deal with the situation when \mathbf{a} is a singleton $\{a\}$, $a \in \mathbb{U}$. In this case, we abbreviate $\theta_{\mathbf{a}}$ as θ_a . Denote by τ the backshift operator defined by

$$(\tau f)(t) = f(t - 1), \quad t \in \mathbb{R}.$$

Lemma 4.2 *If \mathbf{a} consists of a single point a for some $a \in \mathbb{U}$ then a function $f \in L^2(\mathbb{R})$ satisfies (4.2) if and only if*

$$(\tau \hat{f})(\xi) = a \hat{f}(\xi), \quad \text{for all } \xi \leq 0. \quad (4.3)$$

Proof: Let $f \in L^2(\mathbb{R})$. We compute that

$$(fe^{i\theta_a})^\wedge = -a\hat{f} + (1 - |a|^2) \sum_{k \in \mathbb{N}} \bar{a}^{k-1} \tau^k \hat{f}. \quad (4.4)$$

By taking the Fourier transform of both sides of (4.2) and engaging (1.3), we see that (4.2) is equivalent to

$$(fe^{i\theta_a})^\wedge(\xi) = 0, \quad \xi \leq 0.$$

By (4.4), the above equation can be rewritten as

$$-a\hat{f}(\xi) + (1 - |a|^2) \sum_{k \in \mathbb{N}} \bar{a}^{k-1} (\tau^k \hat{f})(\xi) = 0, \quad \xi \leq 0. \quad (4.5)$$

It remains to prove that (4.5) and (4.3) imply each other. Assume that (4.3) holds true. Then we observe for all $\xi \leq 0$ that

$$\begin{aligned} -a\hat{f}(\xi) + (1 - |a|^2) \sum_{k \in \mathbb{N}} \bar{a}^{k-1} \tau^k \hat{f}(\xi) &= -a\hat{f}(\xi) + (1 - |a|^2) \sum_{k \in \mathbb{N}} \bar{a}^{k-1} a^k \hat{f}(\xi) \\ &= a\hat{f}(\xi) \left(-1 + (1 - |a|^2) \sum_{k \in \mathbb{N}} (|a|^2)^{k-1} \right) = 0. \end{aligned}$$

On the other hand, assume that (4.5) holds. We apply τ to the left hand side of (4.5) and then multiply it by \bar{a} to obtain that

$$-|a|^2 (\tau \hat{f})(\xi) + (1 - |a|^2) \sum_{k=2}^{\infty} \bar{a}^{k-1} (\tau^k \hat{f})(\xi) = 0, \quad \xi \leq 0.$$

Subtracting the above equation from (4.5) gives

$$-a\hat{f}(\xi) + |a|^2 (\tau \hat{f})(\xi) + (1 - |a|^2) (\tau \hat{f})(\xi) = 0, \quad \xi \leq 0.$$

An rearrangement of the above equation yields (4.3) and completes the proof. \square

We next present the crucial lemma leading to the characterization of real f satisfying (4.1). To this end, we denote for $\mathbf{a} := (a_j : j \in \mathbb{N}_n) \in \mathbb{U}^n$ by $\tau_{\mathbf{a}}$ the operator

$$(\tau_{\mathbf{a}} f) = \left(\prod_{j \in \mathbb{N}_n} (\tau - a_j) \right) f.$$

Lemma 4.3 *A function $f \in L^2(\mathbb{R})$ satisfies (4.2) if and only if*

$$(\tau_{\mathbf{a}} \hat{f})(\xi) = 0, \quad \text{for all } \xi \leq 0. \quad (4.6)$$

Proof: We shall use the induction on the number n of factors in the Blaschke product $\mathcal{B}_{\mathbf{a}}$. By Lemma 4.2, the result holds true when $n = 1$. Assume that $n \geq 2$ and the result also holds for the $n - 1$ case. Let $\mathbf{a}' := (a_j : 1 \leq j \leq n - 1) \in \mathbb{U}^{n-1}$ and $g = f e^{i\theta_{a_n}}$. Then (4.2) is equivalent to

$$H(g e^{i\theta_{\mathbf{a}'}}) = -i g e^{i\theta_{\mathbf{a}'}}.$$

By induction, the above equation holds true if and only if

$$(\tau_{\mathbf{a}'} \hat{g})(\xi) = 0, \quad \text{for all } \xi \leq 0.$$

We have computed that

$$\hat{g} = -a_n \hat{f} + (1 - |a_n|^2) \sum_{k \in \mathbb{N}} \bar{a}_n^{k-1} \tau^k \hat{f}.$$

Therefore, it suffices to show that (4.6) is equivalent to

$$-a_n \tau_{\mathbf{a}'} \hat{f}(\xi) + (1 - |a_n|^2) \tau_{\mathbf{a}'} \sum_{k \in \mathbb{N}} \bar{a}_n^{k-1} \tau^k \hat{f}(\xi) = 0, \quad \text{for all } \xi \leq 0.$$

Similar arguments as those in the proof of Lemma 4.2 fulfils this purpose. \square

We conclude the results of this section by far into the following theorem. Note that $\iota := \tau^{-1}$ is the shift operator defined as

$$(\iota f)(t) = f(t+1), \quad t \in \mathbb{R}.$$

We introduce another operator $\iota_{\bar{\mathbf{a}}}$ by setting

$$\iota_{\bar{\mathbf{a}}} = \prod_{j \in \mathbb{N}_n} (\iota - \bar{a}_j).$$

Theorem 4.4 *A real function $f \in L^2(\mathbb{R})$ satisfies (4.1) if and only if*

$$(\tau_{\mathbf{a}} \hat{f})(\xi) = 0, \text{ for all } \xi \leq 0 \text{ and } (\iota_{\bar{\mathbf{a}}} \hat{f})(\xi) = 0, \text{ for all } \xi \geq 0. \quad (4.7)$$

Proof: The result follow immediately from Lemmas 4.1, 4.2, 4.3, and the fact that for a real function $f \in L^2(\mathbb{R})$, $\overline{\hat{f}(\xi)} = \hat{f}(-\xi)$, $\xi \in \mathbb{R}$. \square

We next seek a closed form for real functions f satisfying (4.1).

Theorem 4.5 *Let λ_j be the distinct elements in \mathbf{a} with multiplicity n_j , $1 \leq j \leq k$. Then a real function $f \in L^2(\mathbb{R})$ satisfies (4.1) if and only if there exist functions $c_{jl} \in L^2(\mathbb{R})$ with $\text{supp}(c_{jl}) \subseteq [-1, 0]$, $1 \leq j \leq k$, $1 \leq l \leq n_j$ such that*

$$f(x) = \text{Re} \left(\sum_{j=1}^k \sum_{l=1}^{n_j} \frac{c_{jl}(x)}{(1 - \lambda_j e^{-ix})^l} \right). \quad (4.8)$$

Proof: Assume that f is a real function in $L^2(\mathbb{R})$ that satisfies (4.1). Then by Theorem 4.4, \hat{f} satisfies (4.7). It implies that for each $\xi \in (-1, 0]$, $\hat{f}(\xi - m)$ satisfies the difference equation

$$(\tau_{\mathbf{a}} \hat{f})(\xi - m) = 0, \quad m \in \mathbb{Z}_+. \quad (4.9)$$

By the general solution of a difference equation, there exist functions e_{jl} on $(-1, 0]$, $j \in \mathbb{N}_k$, $l \in \mathbb{N}_{n_j}$ such that

$$\hat{f}(\xi - m) = \sum_{j \in \mathbb{N}_k} \sum_{l \in \mathbb{N}_{n_j}} e_{jl}(\xi) \lambda_j^m \prod_{q=1}^{l-1} (m+q), \quad m \in \mathbb{Z}_+. \quad (4.10)$$

Since f is real, we have that

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} d\xi = \frac{1}{\pi} \text{Re} \left(\int_{-\infty}^0 \hat{f}(\xi) e^{ix\xi} d\xi \right) \\ &= \frac{1}{\pi} \text{Re} \left(\sum_{m=0}^{\infty} \int_{-m-1}^{-m} \hat{f}(\xi) e^{ix\xi} d\xi \right) \\ &= \frac{1}{\pi} \text{Re} \left(\sum_{m=0}^{\infty} \int_{-1}^0 \hat{f}(\xi - m) e^{ix(\xi-m)} d\xi \right) \end{aligned} \quad (4.11)$$

Now we apply (4.10) to obtain that

$$\sum_{m=0}^{\infty} \int_{-1}^0 \hat{f}(\xi - m) e^{ix(\xi-m)} d\xi = \sum_{j \in \mathbb{N}_k} \sum_{l \in \mathbb{N}_{n_j}} \int_{-1}^0 \left(\sum_{m=0}^{\infty} \lambda_j^m e^{-imx} \prod_{q=1}^{l-1} (m+q) \right) e_{jl}(\xi) e^{ix\xi} d\xi. \quad (4.12)$$

Recall that

$$\sum_{m=0}^{\infty} \lambda_j^m e^{-imx} \prod_{q=1}^{l-1} (m+q) = \frac{(l-1)!}{(1-\lambda_j e^{-ix})^l}, \quad j \in \mathbb{N}_k, \quad l \in \mathbb{N}_{n_j}. \quad (4.13)$$

Combining (4.11), (4.12), and (4.13) yields that

$$f(x) = \operatorname{Re} \left(\sum_{j \in \mathbb{N}_k} \sum_{l \in \mathbb{N}_{n_j}} \frac{1}{(1-\lambda_j e^{-ix})^l} \frac{(l-1)!}{\pi} \int_{-1}^0 e_{jl}(\xi) e^{ix\xi} d\xi \right).$$

Finally, setting

$$c_{jl}(x) = \frac{(l-1)!}{\pi} \int_{-1}^0 e_{jl}(\xi) e^{ix\xi} d\xi, \quad j \in \mathbb{N}_k, \quad l \in \mathbb{N}_{n_j}$$

leads to the form (4.7) of f .

Conversely, if f is of the form (4.7) then by reversing the above arguments, we obtain (4.10). Thus, for each $\xi \in (-1, 0]$, $\hat{f}(\xi - m)$ satisfies the difference equation (4.9). Therefore, f satisfies (4.6). Since f is real, (4.7) holds true. By Lemma 4.4, f satisfies (4.1). \square

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